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## Citation for published version:

Dreher, M & Schnur, J 2015, 'The combined viscous semi-classical limit for a quantum hydrodynamic system with barrier potential', *Journal of Mathematical Analysis and Applications*, vol. 425, no. 2, pp. 1113-1133. <https://doi.org/10.1016/j.jmaa.2015.01.019>

## Digital Object Identifier (DOI):

[10.1016/j.jmaa.2015.01.019](https://doi.org/10.1016/j.jmaa.2015.01.019)

## Link:

[Link to publication record in Heriot-Watt Research Portal](#)

## Document Version:

Peer reviewed version

## Published In:

Journal of Mathematical Analysis and Applications

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This is the Final Accepted Version of the Manuscript with title *The combined viscous semi-classical limit for a quantum hydrodynamic system with barrier potential* that appeared in *Journal of Mathematical Analysis and Applications*, Volume 425 (2015), pages 1113–1133. It is accessible at  
<http://dx.doi.org/10.1016/j.jmaa.2015.01.019>

# The combined viscous semi-classical limit for a quantum hydrodynamic system with barrier potential

Michael Dreher<sup>1</sup>, Johannes Schnur<sup>2</sup>

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## Abstract

We investigate the viscous model of quantum hydrodynamics, which describes the charge transport in a certain semiconductor. Quantum mechanical effects lead to third order derivatives, turning the stationary system into an elliptic system of mixed order in the sense of Douglis–Nirenberg. In the case most relevant to applications, the semiconductor device features a piecewise constant barrier potential. In the case of thermodynamic equilibrium, we obtain asymptotic expansions of interfacial layers of the particle density in neighbourhoods of the jump points of this barrier potential, and we present rigorous proofs of uniform estimates of the remainder terms in these asymptotic expansions.

2010 Maths Subject Classification: 34E05, 76Y05, 76N20.

Keywords: boundary layers, quantum hydrodynamics, remainder estimates.

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## 1. Introduction

The ongoing miniaturisation of electronical devices requires the investigation of mathematical models for the electron transport that include quantum mechanical terms. One of these models is the isentropic viscous quantum hy-

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hydrodynamic model

$$\begin{cases} \partial_t n - \operatorname{div} J = \nu \Delta n, \\ \partial_t J - \operatorname{div} \left( \frac{J \otimes J}{n} \right) - \nabla p(n) + n \nabla (V + V_B) + \frac{\varepsilon^2}{6} n \nabla \frac{\Delta \sqrt{n}}{\sqrt{n}} = \nu \Delta J - \frac{J}{\tau}, \\ \lambda^2 \Delta V = n - \mathcal{C}, \end{cases} \quad (1)$$

formulated for the unknown functions  $(n, J, V)$ , and the independent variables are  $t \in \mathbb{R}$  as time, and  $x \in \mathbb{R}^d$  as space. The unknown functions are the (positive) scalar electron density  $n$ , the vectorial electric current density  $J$ , and the scalar electric potential  $V$ . The item  $p(n)$  is a generic pressure term, and a common choice is  $p(n) = Tn + \mu n$ , with a temperature  $T$  given by a relation  $T(n) = T_0 n^{\gamma-1}$  for a positive constant  $T_0$  and some  $\gamma \geq 1$ , and  $\mu > 0$ . Furthermore, the barrier potential  $V_B = V_B(x)$  and the doping profile  $\mathcal{C} = \mathcal{C}(x)$  of the semiconductor are given functions that describe certain material properties; these two functions are typically piecewise constant, and they are of crucial importance for the working principle of devices as the resonant tunnel diode. The purpose of this paper is to study analytically the behaviour of the solutions  $(n, J, V)$  near the jump points of the barrier potential  $V_B$ .

Additionally, we have certain positive physical constants, which have been scaled for ease of notation: The Planck constant  $\varepsilon$ , a relaxation time  $\tau$ , the Debye length  $\lambda$ , and a viscosity constant  $\nu$ .

A model (1) without the viscosity terms on the right hand side was proposed in [11] as a variant of the classical Euler–Poisson system, augmented by a term

$$\frac{\varepsilon^2}{6} n \nabla \frac{\Delta \sqrt{n}}{\sqrt{n}} =: \frac{\varepsilon^2}{6} n \nabla B(n)$$

that involves the Bohm potential  $B(n)$  and describes quantum mechanical effects. The expectation is that this term is negligible in those regions where the electron flow can be described in terms of classical physics (i.e., in some regions far away from jump points of  $V_B$ ).

There are various ways to derive (inviscid) quantum hydrodynamic models; we mention the traditional moment method applied to the collision Wigner equation [11], an approach via WKB wave functions from the Schrödinger Poisson system [14], and the entropy minimization approach [16]. Augmenting the Wigner equation with a Fokker Planck operator that describes the interaction of the electrons with the phonons of the crystal lattice, the dissipation terms  $\nu \Delta n$  and  $\nu \Delta J$  appear, see [3]. For an overview of this field, we refer to [1] and [15].

The quantum mechanical effects enter the system mainly via the Bohm term  $B(n)$ , which introduces third order spatial derivatives into the momentum balance equation, which complicates analytical studies of (1) considerably, compare [4], [5], [6], [10] for results on the transient problem without barrier potential. Further analytical difficulties arise from the barrier potential  $V_B$  having jumps, and in that situation the second equation of (1) must be understood in the

distributional sense. We are not aware of any analytical results concerning the transient system (1), however we mention numerical simulations in [8], [11], [13], [17], [18], and [19].

We focus our attention to a one-dimensional, stationary system,

$$\left\{ \begin{array}{l} J' = -\nu n'' \quad \text{in } [0, 1], \\ 2\varepsilon^2 n \left( \frac{\sqrt{n''}}{\sqrt{n}} \right)' - \nu J'' - (p(n))' + \frac{J}{\tau} = \left( \frac{J^2}{n} \right)' - n(V + V_B)' \quad \text{in } [0, 1], \\ \lambda^2 V'' = n - \mathcal{C} \quad \text{in } [0, 1]. \end{array} \right. \quad (2)$$

For such a stationary system (without barrier potential), the existence of solutions was shown in [12], assuming small applied voltages  $V(1) - V(0)$  and small currents  $J$ , which corresponds to a subsonic condition for the moving electrons. Although formulated for the isothermal case  $p(n) = (T_0 + \mu)n$ , the results of [12] seem to generalize to the case of general pressure terms  $p(n)$ . And we also mention [9], where it was shown (in the isothermal case) that solutions  $(n, V, J) \in W^{2,2}(0, 1) \times W^{2,2}(0, 1) \times W^{1,2}(0, 1)$  to (2) for given (possibly large) Dirichlet boundary values for  $V$  and periodic boundary values for  $n$  do exist.

The purpose of the present paper is to extend the solution theory of [9] towards an asymptotic expansion of the solution, for vanishing values of the quantum mechanical parameters  $\varepsilon$  and  $\nu$ , focussing on the equilibrium case. See also [20] for further results. It turns out that we find a similar asymptotic expansion of the particle density  $n$  as in [2], [7] for a stationary quantum drift diffusion model.

The solution theory in [9] is based on a reformulation of the system (2) by means of a viscosity-adjusted Fermi level

$$\left\{ \begin{array}{l} F = -(V + V_B) + h(n) + \frac{\nu}{\tau} \ln(n) - 2(\varepsilon^2 + \nu^2) \frac{\sqrt{n''}}{\sqrt{n}}, \\ nF' = -\left( \frac{J_0^2}{n} \right)' + 2J_0\nu \left( 2\frac{\sqrt{n''}}{\sqrt{n}} - \frac{(n')^2}{2n^2} \right) + \frac{J_0}{\tau}, \end{array} \right.$$

where the electric current was replaced by the identity  $J = -\nu n' + J_0$  for some constant of integration  $J_0$ , and where  $h: (0, \infty) \rightarrow \mathbb{R}$  is the enthalpy to  $p$ , satisfying  $sh'(s) = p'(s)$  ( $s > 0$ ). This reformulation reveals the characteristic parameter  $\kappa = \varepsilon^2 + \nu^2$ , which is the coefficient of the derivative of highest order. It is expectable that the solutions  $n = n_\kappa$  depend significantly on  $\kappa$ . Even in the one-dimensional setting, it seems to be a delicate matter whether the solutions  $n_\kappa$  converge to a limiting function  $n_0$  as  $\kappa$  tends to zero, and uniform pointwise bounds for the electron densities from above and away from zero are not known to hold (except in the equilibrium case). In this paper, we consider the thermal equilibrium case of the Fermi levels, which refers to the case of  $F \equiv \text{const}$  and  $J_0 = 0$ . We also assume the physically reasonable situation where  $V_B + V_\kappa$  vanishes at the endpoints of the interval  $[0, 1]$ . By a

straightforward generalisation of the approach of [9], the known results read as follows, formulated in terms of  $u := \sqrt{n}$ :

**Theorem 1.1** ([9],[20]). *Let  $\varepsilon, \lambda, \nu, \tau > 0$ , suppose  $\mathcal{C}, V_B \in L^\infty(0, 1)$ , and assume that the Fermi level is a constant function  $F \in \mathbb{R}$ . Let  $h$  be the enthalpy to the smooth and strictly monotonically increasing pressure term  $p$ , which fulfills  $sh'(s) = p'(s)$ ,  $s > 0$ , and additionally assume*

$$(i) \lim_{s \rightarrow 0} h(s) = -\infty \text{ and } \lim_{s \rightarrow \infty} h(s) = \infty.$$

$$(ii) s \mapsto \sqrt{s}h(s) \text{ is continuous in } [0, \infty) \text{ and } s \mapsto \frac{\ln(s)}{h(s)} \text{ is continuous in } (0, \infty).$$

$$(iii) \text{ For any positive } f \in W^{1,2}(0, 1), \text{ there holds } h(f), \frac{\ln(f)}{h(f)} \in W^{1,2}(0, 1) \text{ with the chain rule being valid.}$$

Then, for any  $\kappa := \varepsilon^2 + \nu^2$ , there exists a solution  $(u_\kappa, V_\kappa) \in W^{2,2}(0, 1) \times W^{2,2}(0, 1)$  to the system of equations

$$\begin{cases} 2\kappa u_\kappa'' = -\left(F + V_B + V_\kappa - h(u_\kappa^2) - \frac{\nu}{\tau} \ln(u_\kappa^2)\right) u_\kappa, & \text{in } [0, 1], \\ \lambda^2 V_\kappa'' = u_\kappa^2 - \mathcal{C}, & \text{in } [0, 1], \\ V_\kappa(0) + V_B(0) = 0, & V_\kappa(1) + V_B(1) = 0, \end{cases} \quad (3)$$

for homogeneous Neumann boundary conditions  $u_\kappa'(0) = u_\kappa'(1) = 0$  and periodic boundary conditions  $u_\kappa(0) = u_\kappa(1)$ ,  $u_\kappa'(0) = u_\kappa'(1)$ , respectively. Moreover, for any total mass  $C^* > 0$  there exist solutions  $(u_\kappa, V_\kappa) \in W^{2,2}(0, 1) \times W^{2,2}(0, 1)$ ,  $\beta_\kappa \in \mathbb{R}$ , to the system of equations

$$\begin{cases} 2\kappa u_\kappa'' = -\left(F + \beta_\kappa + V_B + V_\kappa - h(u_\kappa^2) - \frac{\nu}{\tau} \ln(u_\kappa^2)\right) u_\kappa, & \text{in } [0, 1], \\ \lambda^2 V_\kappa'' = u_\kappa^2 - \mathcal{C}, & \text{in } [0, 1], \\ \int_0^1 u_\kappa^2 = C^*, \\ V_\kappa(0) + V_B(0) = 0, & V_\kappa(1) + V_B(1) = 0, \end{cases} \quad (4)$$

for homogeneous Neumann boundary conditions for  $u_\kappa$  and periodic boundary conditions for  $u_\kappa$ , respectively. The functions  $n_\kappa = u_\kappa^2$ ,  $V_\kappa$  and  $J_\kappa = -\nu n_\kappa'$  form a solution to the viscous quantum hydrodynamic system (2) and admit the respective boundary values. There exist constants  $C_0, \dots, C_4 > 0$  such that, for  $0 < \kappa < \kappa_0$ ,

$$\begin{aligned} C_0 &\leq u_\kappa \leq C_1, & \|V_\kappa\|_{C^1(0,1)} &\leq C_2, & |\beta_\kappa| &\leq C_3, \\ \|u_\kappa'\|_{L^\infty(0,1)} &\leq C_4 \kappa^{-1/2}. \end{aligned}$$

## 2. Statement of the problem and main result

Throughout the paper, we always assume that the situation of Theorem 1.1 is given and that  $u_\kappa, V_\kappa$  are corresponding solutions. In case that the total

mass  $C^*$  is prescribed for  $u_\kappa^2$ , let  $\beta_\kappa$  be the corresponding Lagrange multiplier appearing in (4). We investigate the behavior of solutions as  $\kappa$  tends to zero for the case of piecewise constant barrier potentials  $V_B$ , which is the physically most relevant case. Formally letting  $\kappa = 0$  in (4), we expect that potentially existing limiting functions  $u_0$  and  $V_0$  fulfill the identities

$$0 = -(F + \beta_0 + V_B + V_0 - h(u_0^2))u_0 \quad \text{and} \quad \lambda^2 V_0'' = u_0^2 - \mathcal{C}. \quad (5)$$

The first limiting equation, however, shows that the expected limit  $u_0$  will jump at the jump points of  $V_B$ , and therefore, convergence of  $(u_\kappa)_{\kappa \rightarrow 0}$  to  $u_0$  is not possible in strong topologies like  $L^\infty(0, 1)$ . However, convergence of the sequence  $(V_\kappa)_{\kappa \rightarrow 0}$  in  $W^{1,2}(0, 1)$  will be shown by monotonicity arguments; and as a consequence we also obtain  $L^p$  convergence of the sequence  $(u_\kappa)_{\kappa \rightarrow 0}$ , which is locally uniform away from the jump points of  $V_B$ . Our first main theorem reads as follows:

**Theorem 2.1.** *Let the situation of Theorem 1.1 be given and assume that the barrier potential  $V_B$  is a piecewise constant function. Then there exist  $V_0 \in W^{2,2}(0, 1)$ ,  $u_0 \in L^\infty(0, 1)$  and (if appropriate)  $\beta_0 \in \mathbb{R}$  solving (5) and the Dirichlet boundary conditions for  $V$  such that*

$$\|V_0 - V_\kappa\|_{W^{1,2}(0,1)} \leq C\kappa^{1/4}, \quad (6)$$

$$|\beta_0 - \beta_\kappa| \leq C\kappa^{1/4}, \quad (7)$$

$$\|u_0^2 - u_\kappa^2\|_{L^p(0,1)} \leq C\kappa^{1/4p} \quad (0 < \kappa < \kappa_0). \quad (8)$$

Moreover, for any subinterval  $[s_0, s_1] \subset [0, 1]$  of length  $L := s_1 - s_0$ , where  $V_B$  is constant, it holds

$$\|u_0^2 - u_\kappa^2\|_{L^\infty(s_0+L\kappa^{1/4}, s_1-L\kappa^{1/4})} \leq C\kappa^{1/4}. \quad (9)$$

Since all solutions  $u_\kappa$  ( $\kappa > 0$ ) are continuously differentiable by the Sobolev embedding theorem, and the sequence  $(u_\kappa)_{\kappa \rightarrow 0}$  converges uniformly in the interior of subintervals, where  $V_B$  is constant, c.f. (9), the functions  $u_\kappa$  are expected to form interface layers near the jump points of  $V_B$ . The quantum term  $\kappa u_\kappa''$  has a non-small value only in this layer regime, and it is natural to expect a layer width of order  $\mathcal{O}(\kappa^{1/2})$ . Our second main theorem makes this statement rigorous, by means of an analytically proven remainder estimate:

**Theorem 2.2 (Zeroth order asymptotic expansion).** *Let the situation of Theorem 2.1 be given. Let  $s_1 = 0$ ,  $s_{N+1} = 1$  and  $s_2, \dots, s_N$  be the jump points of  $V_B$ . There exist  $W_\kappa: [0, 1] \rightarrow \mathbb{R}$  and a positive function  $c_0: [0, 1] \rightarrow (0, \infty)$  which fulfills  $c_0(x) = u_0(s_i \pm)$ , ( $i = 1, \dots, N+1$ ), in half-sided neighbourhoods of  $s_i$ , such that*

$$\left\| u_\kappa - u_0 \frac{W_\kappa}{c_0} \right\|_{L^2(0,1)} + \|V_\kappa - V_0\|_{W^{1,2}(0,1)} \leq C\kappa^{1/2} \quad (10)$$

and

$$\left\| u_\kappa - u_0 \frac{W_\kappa}{c_0} \right\|_{L^\infty(0,1)} \leq C\kappa^{1/4}. \quad (11)$$

Near any jump point  $s_i$  of  $V_B$ , the functions  $W_\kappa$  locally admit a representation

$$W_\kappa(x) = w\left(\frac{x - s_i}{\kappa^{1/2}}\right)$$

for a function  $w \in C^1(\mathbb{R})$  with  $\lim_{y \rightarrow \pm\infty} w(y) = u_0(s_i \pm)$  and exponential convergence to both limits.

The proofs of both results rely on the spatial dimension being one — we use Sobolev’s embedding theorem and ODE techniques extensively.

The structure of the paper is as follows. In Section 3, we show various bounds on derivatives of  $u_\kappa$ , and the key result is (17), which shows that the Bohm potential term  $B(u_\kappa^2)$  is indeed negligible in the exterior region, which is, by convention, “far away” from the jumps of  $V_B$ . Then Theorem 2.1 is proved in Section 4 by means of monotonicity principles. This gives us the first term  $u_0$  of the asymptotic expansion of  $u_\kappa$  in the exterior region. Section 5 contains results on the asymptotic expansion of  $u_\kappa$  in a certain interior region (which is “near the jumps of  $V_B$ ”), and on the matching of both asymptotic expansions, see Lemmas 5.3 and 5.5. Choosing a different set of multipliers, we then improve the remainder estimates in Section 6, concluding the proof of Theorem 2.2.

**Acknowledgements.** The research of the first author has been supported by a DFG project (446 CHV 113/170/0-2), and both authors are grateful to the Center of Evolution Equations of the University of Konstanz for support. We thank the referee for useful remarks that helped to improve an earlier version of the manuscript.

### 3. First estimates to solutions

In the thermal equilibrium case, we have  $J_0 = 0$  and a constant Fermi level  $F \in \mathbb{R}$ , so that  $J = -\nu n'$ . Using this and  $\frac{\sqrt{n''}}{\sqrt{n}} = \frac{n''}{2n} - \frac{n'^2}{n^2}$ , the weak formulation of the second equation of (2) reads as

$$\begin{aligned} & - \int_0^1 \left( p'(n) + \frac{\nu}{\tau} \right) n' \varphi - (\varepsilon^2 + \nu^2) \int_0^1 \left( \frac{n'^2}{n} \right)' \varphi - (\varepsilon^2 + \nu^2) \int_0^1 n'' \varphi' \\ & + \int_0^1 n V' \varphi - \int_0^1 V_B (n \varphi)' = 0 \quad (\varphi \in C_c^\infty(0, 1)). \end{aligned} \quad (12)$$

**Lemma 3.1 (Basic exterior estimates to solutions).**

*In the situation of Theorem 1.1, assume that  $V_B$  is constant in some non-trivial*



interval  $[s_0, s_1] \subset (0, 1)$ . Then, for  $C^* = \int_0^1 n \, dx$ , the estimate

$$\begin{aligned} K_0 \int_{s_0+\sigma L}^{s_1-\sigma L} u'^2 \, dx + (\varepsilon^2 + \nu^2) \int_{s_0+\sigma L}^{s_1-\sigma L} u''^2 + \frac{u'^4}{24u^2} \, dx \\ \leq \frac{(\varepsilon^2 + \nu^2)CC^*}{\sigma^4 L^4} + \frac{CC^*}{K_0 \lambda^4}, \quad 0 < \sigma < \frac{1}{2}, \end{aligned} \quad (13)$$

holds, where  $K_0$  only depends on  $C_0, C_1, p$ ; and  $C$  only depends on  $\|\mathcal{C}\|_{L^\infty(0,1)}$ . Additionally,  $L := s_1 - s_0$  is the length of the interval  $[s_0, s_1]$ . Consequently, for  $I_\kappa := [s_0 + \kappa^{1/4}L, s_1 - \kappa^{1/4}L]$ , there holds

$$\|u'\|_{L^2(I_\kappa)} \leq C, \quad (14)$$

$$\|\kappa u''\|_{L^2(I_\kappa)} \leq C\kappa^{1/2}, \quad (15)$$

$$\|u'\|_{L^\infty(I_\kappa)} \leq C\kappa^{-1/4}, \quad (16)$$

$$\left\| 2\kappa \frac{u''}{u} \right\|_{L^\infty(I_\kappa)} \leq C\kappa^{1/4}, \quad (17)$$

with some constant  $C$  which does not depend on  $0 < \kappa < \kappa_0$ .

*Proof.* Let  $a_0 := \frac{1}{\sigma^4 L^4}$  and define  $\psi \in W_0^{1,2}(0, 1)$  by

$$\psi(x) := \begin{cases} a_0(x - s_0)^4, & s_0 \leq x \leq s_0 + \sigma L, \\ 1, & s_0 + \sigma L \leq x \leq s_1 - \sigma L, \\ a_0(s_1 - x)^4, & s_1 - \sigma L \leq x \leq s_1, \\ 0, & x \notin [s_0, s_1]. \end{cases}$$

Using  $\varphi := \frac{u'}{u}\psi$  as a test function in (12), we obtain in terms of  $u = \sqrt{n}$

$$\begin{aligned} 2 \int_0^1 \left( p'(u^2) + \frac{\nu}{\tau} \right) u'^2 \psi \, dx \\ + 2(\varepsilon^2 + \nu^2) \int_0^1 2(u'^2)' \frac{u'}{u} \psi + (u''u + u'^2) \left( \frac{u'}{u} \psi \right)' \, dx \\ = \int_0^1 uu'V'\psi \, dx, \end{aligned}$$

since  $V_B$  is constant on  $\text{supp } \psi$ . We abbreviate this identity by

$$I_1 + 2(\varepsilon^2 + \nu^2)I_2 = J_1.$$

By assumption, we have  $p'(\xi) > 0$  for  $\xi > 0$  and  $C_0^2 \leq u^2(x) \leq C_1^2$  for  $x \in [0, 1]$  from Theorem 1.1, so that  $p'(u^2(x)) \geq K_0 > 0$  for all  $x \in [0, 1]$  and some  $K_0$ . Then,

$$2 \left( K_0 + \frac{\nu}{\tau} \right) \int_0^1 u'^2 \psi \, dx \leq I_1.$$

Re-ordering terms in  $I_2$ , we find

$$I_2 = \int_0^1 \left( u''^2 + 4 \frac{u'' u'^2}{u} - \frac{u'^4}{u^2} \right) \psi + \left( u' u'' + \frac{u'^3}{u} \right) \psi' dx. \quad (18)$$

An integration by parts yields

$$0 = \int_0^1 \left( 3 \frac{u'' u'^2}{u} - \frac{u'^4}{u^2} \right) \psi + \frac{u'^3}{u} \psi' dx. \quad (19)$$

Now we form  $(18) - \frac{4}{3}(19)$ , and the result is

$$\begin{aligned} I_2 &= \int_0^1 \left( u''^2 + \frac{1}{3} \frac{u'^4}{u^2} \right) \psi + u'' u' \psi' - \frac{1}{3} \frac{u'^3}{u} \psi' dx \\ &=: \int_0^1 \left( u''^2 + \frac{1}{3} \frac{u'^4}{u^2} \right) \psi dx + I_{2,1} + I_{2,2}. \end{aligned}$$

Because  $|\psi'| \leq \frac{4}{\sigma L} \psi^{3/4}$ , exploiting Young's inequality with exponents 2, 4, 4 gives

$$\begin{aligned} |I_{2,1}| &\leq \int_0^1 \left| u'' \psi^{1/2} \right| \cdot \left| \frac{u' \psi^{1/4}}{\sqrt{2} u^{1/2}} \right| \cdot \frac{4\sqrt{2} u^{1/2}}{\sigma L} dx \\ &\leq \int_0^1 \frac{1}{2} u''^2 \psi dx + \int_0^1 \frac{u'^4}{16 u^2} \psi dx + \int_0^1 \frac{256 u^2}{\sigma^4 L^4} \chi_{[s_0, s_1]} dx. \end{aligned}$$

Using Young's inequality with the exponents  $\frac{4}{3}$  and 4, we further obtain

$$|I_{2,2}| \leq \frac{1}{3} \int_0^1 \frac{(u'^4 \psi)^{3/4}}{u^{3/2}} \cdot \frac{4 u^{1/2}}{\sigma L} dx \leq \int_0^1 \frac{u'^4}{4 u^2} \psi + \frac{64 u^2}{3 \sigma^4 L^4} \chi_{[s_0, s_1]} dx.$$

Since  $\int_0^1 u^2 dx = C^*$ , we infer

$$\int_0^1 \frac{1}{2} u''^2 \psi dx + \left( \frac{1}{3} - \frac{1}{16} - \frac{1}{4} \right) \int_0^1 \frac{u'^4}{u^2} \psi dx \leq I_2 + \frac{256 C^*}{\sigma^4 L^4} + \frac{64 C^*}{3 \sigma^4 L^4}.$$

Thus,

$$(\varepsilon^2 + \nu^2) \int_0^1 \left( u''^2 + \frac{1}{24} \frac{u'^4}{u^2} \right) \psi dx \leq 2(\varepsilon^2 + \nu^2) I_2 + \frac{2(\varepsilon^2 + \nu^2) C C^*}{\sigma^4 L^4}.$$

Concerning the right hand side  $J_1$ , we estimate

$$|J_1| \leq K_0 \int_0^1 u'^2 \psi dx + \frac{C \|V'\|_{L^\infty(0,1)}^2 C^*}{K_0} \leq K_0 \int_0^1 u'^2 \psi dx + \frac{C C^*}{\lambda^4 K_0}.$$

Combining all estimates, inequality (13) follows. For  $\sigma = \kappa^{1/4}$ , this immediately yields inequalities (14) and (15); and by interpolation we also obtain (16). As

$\frac{u''}{u}$  is smooth in  $I_\kappa$ , we find

$$\begin{aligned} 2\kappa \left( \frac{u''}{u} \right)' &= - \left( F + V_B + V + \beta - h(u^2) - \frac{\nu}{\tau} \ln(u^2) \right)' \\ &= -V' + 2 \left( p'(u^2) + \frac{\nu}{\tau} \right) \frac{u'}{u} \end{aligned}$$

as an equality in  $I_\kappa$ . The right hand side is uniformly bounded in  $L^2(I_\kappa)$  by inequality (14), the pointwise upper and lower bounds to  $u$  and the uniform boundedness of  $\|V\|_{W^{2,2}(0,1)}$ . Joining this bound with inequality (15), we obtain

$$\left\| 2\kappa \frac{u''}{u} \right\|_{W^{1,2}(I_\kappa)} \leq C.$$

Interpolating this inequality with estimate (15), inequality (17) follows.  $\square$

#### 4. Exterior convergence results

Estimate (17) already shows that the quantum mechanical Bohm term  $\kappa \frac{u''}{u_\kappa}$  decays in the interior of subintervals, where  $V_B$  is constant. We are now in the position to prove the convergence of  $(V_\kappa)_{\kappa \rightarrow 0}$  and consequently, convergence of  $(u_\kappa)_{\kappa \rightarrow 0}$  also follows.

*Proof of Theorem 2.1.* Let  $I^1, \dots, I^N$  be the maximal intervals in which  $V_B$  is constant and denote by  $I_\kappa^1, \dots, I_\kappa^N$  the corresponding subintervals introduced in Lemma 3.1. By assumption,  $h^{-1}: \mathbb{R} \rightarrow (0, \infty)$  exists and  $u_\kappa^2$  can be expressed by

$$u_\kappa^2 = h^{-1}(F + V_B + V_\kappa + \beta_\kappa) + r_\kappa,$$

where

$$r_\kappa := h^{-1} \left( F + V_B + V_\kappa + \beta_\kappa + 2\kappa \frac{u_\kappa''}{u_\kappa} - \frac{\nu}{\tau} \ln(u_\kappa^2) \right) - h^{-1}(F + V_B + V_\kappa + \beta_\kappa).$$

By Lipschitz continuity of  $h^{-1}$ ,

$$|r_\kappa(x)| \leq C \left( \left| 2\kappa \frac{u_\kappa''(x)}{u_\kappa(x)} \right| + \kappa^{1/2} \right) \quad (x \in (0, 1)),$$

which implies together with inequalities (15) and (17)

$$\|r_\kappa\|_{L^2(I_\kappa)} \leq C\kappa^{1/2}, \tag{20}$$

$$\|r_\kappa\|_{L^\infty(I_\kappa)} \leq C\kappa^{1/4}, \tag{21}$$

for  $0 < \kappa < \kappa_0$ . Let  $0 < \kappa_2 \leq \kappa_1 < \kappa_0$ , abbreviate  $u_i := u_{\kappa_i}$ ,  $V_i := V_{\kappa_i}$ ,  $\beta_i := \beta_{\kappa_i}$ ,  $r_i := r_{\kappa_i}$  and define  $\tilde{V}_i := V_i + \beta_i$  for  $i = 1, 2$ . In the following, we

may formally consider  $\beta_i = 0$  for  $i = 1, 2$  if the additional constraint  $\int u_\kappa^2 = C^*$  is not demanded for the solutions. An integration by parts yields

$$\begin{aligned} \lambda^2 \int_0^1 (V_1' - V_2')^2 dx &= - \int_0^1 (u_1^2 - u_2^2) (\tilde{V}_1 - \tilde{V}_2) dx \\ &= - \sum_{i=1}^N \int_{I^i \setminus I_{\kappa_1}^i} (u_1^2 - u_2^2) (\tilde{V}_1 - \tilde{V}_2) dx \\ &\quad - \sum_{i=1}^N \int_{I_{\kappa_1}^i} (u_1^2 - u_2^2) (\tilde{V}_1 - \tilde{V}_2) dx \\ &=: S_1 + S_2. \end{aligned}$$

Using the uniform pointwise boundedness of  $u_\kappa^2$ , Young's inequality, the Sobolev embedding  $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1)$  and Poincaré's inequality, we find

$$\begin{aligned} S_1 &\leq C \sum_{i=1}^N \|\tilde{V}_1 - \tilde{V}_2\|_{L^\infty(I^i \setminus I_{\kappa_1}^i)} \kappa_1^{1/4} \\ &\leq C \gamma^2 \|\tilde{V}_1 - \tilde{V}_2\|_{W^{1,2}(0,1)}^2 + \gamma^{-2} \kappa_1^{1/2} \\ &\leq K_1 \gamma^2 \left( \|V_1' - V_2'\|_{L^2(0,1)}^2 + |\beta_1 - \beta_2|^2 \right) + \gamma^{-2} \kappa_1^{1/2}, \end{aligned}$$

where  $K_1 > 0$  is a constant und  $\gamma > 0$  is a free parameter which will be chosen later on. Concerning  $S_2$ , we calculate

$$\begin{aligned} &- \sum_{i=1}^N \int_{I_{\kappa_1}^i} (u_1^2 - u_2^2) (\tilde{V}_1 - \tilde{V}_2) dx \\ &= - \sum_{i=1}^N \int_{I_{\kappa_1}^i} \left( h^{-1}(F + V_B + \tilde{V}_1) - h^{-1}(F + V_B + \tilde{V}_2) \right) (\tilde{V}_1 - \tilde{V}_2) dx \\ &\quad - \sum_{i=1}^N \int_{I_{\kappa_1}^i} (r_1 - r_2) (\tilde{V}_1 - \tilde{V}_2) dx. \end{aligned}$$

Now we certainly find a positive number  $K_2$  with

$$\frac{1}{K_2} \leq h'(s^2) \leq K_2, \quad C_0 \leq s \leq C_1. \quad (22)$$

Then  $(h^{-1})'$  enjoys the same bounds, and we get

$$\begin{aligned} &- \sum_{i=1}^N \int_{I_{\kappa_1}^i} \left( h^{-1}(F + V_B + \tilde{V}_1) - h^{-1}(F + V_B + \tilde{V}_2) \right) (\tilde{V}_1 - \tilde{V}_2) dx \\ &\leq -K_2^{-1} \sum_{i=1}^N \|\tilde{V}_1 - \tilde{V}_2\|_{L^2(I_{\kappa_1}^i)}^2. \end{aligned}$$

The Cauchy-Schwarz inequality in combination with both the uniform boundedness of  $(\tilde{V}_\kappa)_{0 < \kappa < \kappa_0}$  in  $L^2(0, 1)$  and estimate (20) implies

$$\begin{aligned} -\sum_{i=1}^N \int_{I_{\kappa_1}^i} (r_1 - r_2) (\tilde{V}_1 - \tilde{V}_2) \, dx &\leq \sum_{i=1}^N \|r_1 - r_2\|_{L^2(I_{\kappa_1}^i)} \|\tilde{V}_1 - \tilde{V}_2\|_{L^2(I_{\kappa_1}^i)} \\ &\leq K_3 \kappa_1^{1/2}, \end{aligned}$$

and we conclude that

$$S_2 \leq -K_2^{-1} \sum_{i=1}^N \|\tilde{V}_1 - \tilde{V}_2\|_{L^2(I_{\kappa_1}^i)}^2 + K_3 \kappa_1^{1/2}.$$

Combining all estimates, we obtain

$$\begin{aligned} (\lambda^2 - K_1 \gamma^2) \|V_1' - V_2'\|_{L^2(0,1)}^2 & \tag{23} \\ &\leq K_1 \gamma^2 |\beta_1 - \beta_2|^2 - K_2^{-1} \sum_{i=1}^N \|\tilde{V}_1 - \tilde{V}_2\|_{L^2(I_{\kappa_1}^i)}^2 + \gamma^{-2} \kappa_1^{1/2} + K_3 \kappa_1^{1/2}. \end{aligned}$$

Let  $\delta > 0$  be a parameter to be determined later on. For small  $\kappa_1$ , we certainly have  $\sum_{i=1}^N \text{meas}(I_{\kappa_1}^i) \geq \frac{1}{2}$ , and then we may estimate

$$\begin{aligned} \frac{1}{2} \delta |\beta_1 - \beta_2|^2 &\leq \delta \sum_{i=1}^N \|\beta_1 - \beta_2\|_{L^2(I_{\kappa_1}^i)}^2 & \tag{24} \\ &\leq 2\delta \sum_{i=1}^N \|\tilde{V}_1 - \tilde{V}_2\|_{L^2(I_{\kappa_1}^i)}^2 + 2\delta K_4 \|V_1' - V_2'\|_{L^2(0,1)}^2 \end{aligned}$$

with some constant  $K_4 > 0$  arising from the Poincaré inequality. Adding estimates (23) and (24), the inequality

$$\begin{aligned} (\lambda^2 - K_1 \gamma^2 - 2\delta K_4) \|V_1' - V_2'\|_{L^2(0,1)}^2 &+ \left( \frac{\delta}{2} - K_1 \gamma^2 \right) |\beta_1 - \beta_2|^2 \\ &\leq (2\delta - K_2^{-1}) \sum_{i=1}^N \|\tilde{V}_1 - \tilde{V}_2\|_{L^2(I_{\kappa_1}^i)}^2 + \gamma^{-2} \kappa_1^{1/2} + K_3 \kappa_1^{1/2} \end{aligned}$$

follows. Choosing  $\delta$  and  $\gamma$  sufficiently small, one obtains

$$\|V_1' - V_2'\|_{L^2(0,1)}^2 + |\beta_1 - \beta_2|^2 \leq C \kappa_1^{1/2},$$

which yields inequalities (6) and (7). To prove the convergence results on  $u_\kappa$ , observe that

$$u_0^2 := h^{-1}(F + V_B + V_0 + \beta_0) \in L^\infty(0, 1)$$

is a positive function on  $[0, 1]$ . For the function

$$k_\nu := h + \frac{\nu}{\tau} \ln$$

it is easily seen that  $\|h^{-1} - k_\nu^{-1}\|_{L^\infty(k_{\nu_0}(I))} \leq C\nu$  for small  $0 < \nu < \nu_0$  and an interval  $I = [a, b] \subset (0, \infty)$ . Since  $V_\kappa$ ,  $\beta_\kappa$  and  $2\kappa \frac{u''_\kappa}{u_\kappa}$  are uniformly bounded in  $L^\infty(I_\kappa^i)$  for  $0 < \kappa < \kappa_0$  and  $i = 1, \dots, N$ , the local Lipschitz continuity of  $k_\nu^{-1}$ , the convergence results for  $V_\kappa$  and  $\beta_\kappa$  and inequality (17) readily yield

$$\|u_0^2 - u_\kappa^2\|_{L^\infty(I_\kappa^i)} \leq C\kappa^{1/4}.$$

From this, the  $L^p$ -estimates for  $u_\kappa - u_0$  are easily obtained by a trivial estimate of the appearing integrals over the regimes outside the subintervals  $I_\kappa^1, \dots, I_\kappa^N$ , because the functions  $u_\kappa$  are uniformly bounded.  $\square$

## 5. Derivation of the zeroth order asymptotic expansion

We now derive differential equations describing the functions  $W_\kappa$  from Theorem 2.2 locally at any jump point  $s_0$  of  $V_B$ . As a preliminary step, we need to show that the derivatives  $u'_\kappa(s_0)$  are not too small — they are of order  $\kappa^{-1/2}$ .

**Lemma 5.1.** *In the situation of Theorem 1.1, assume that  $V_B$  is piecewise constant, and let  $s_0$  be a jump point of  $V_B$ . Then there exists a constant  $C_5 > 0$  such that*

$$C_4\kappa^{-1/2} \geq \|u'_\kappa\|_{L^\infty(0,1)} \geq |u'_\kappa(s_0)| \geq C_5\kappa^{-1/2}, \quad (25)$$

for  $0 < \kappa < \kappa_0$ . We also have, for such  $\kappa$ ,

$$u_\kappa^2(s_0) = \frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{V_B(s_0+) - V_B(s_0-)} + \mathcal{O}(\kappa^{1/4}). \quad (26)$$

**Remark 5.2.** *We remark that  $u_\kappa^2(s_0)$  is (up to an error of size  $\kappa^{1/4}$ ) between the left and right limits  $u_0^2(s_0-)$  and  $u_0^2(s_0+)$ , because the extended mean value theorem gives us*

$$\frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{V_B(s_0+) - V_B(s_0-)} = \frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{h(u_0^2(s_0+)) - h(u_0^2(s_0-))} = \frac{p'(\xi)}{h'(\xi)} = \xi,$$

for some  $\xi$  between  $u_0^2(s_0+)$  and  $u_0^2(s_0-)$ .

*Proof of Lemma 5.1.* We choose, for  $z > 0$ ,

$$K(z) = zk(z) - p(z) - \frac{\nu}{\tau}z, \quad (27)$$

$$H(z) = zh(z) - p(z) \quad (28)$$

as primitive functions of  $k$  and  $h$ , respectively. We may unite the differential equations for  $u_\kappa$  from (3) and (4) into the equation

$$2\kappa u_\kappa'' = -(F + \beta_\kappa + V_B + V_\kappa - k(u_\kappa^2))u_\kappa, \quad (29)$$

tacitly making the convention  $\beta_\kappa = 0$  in the case without mass balance. Then (6) and (7) imply

$$\begin{aligned} -2\kappa u_\kappa'' &= (F + \beta_0 + V_B + V_0 - k(u_\kappa^2))u_\kappa + (\beta_\kappa - \beta_0 + V_\kappa - V_0)u_\kappa \\ &= (h(u_0^2) - k(u_\kappa^2))u_\kappa + \mathcal{O}(\kappa^{1/4}), \end{aligned}$$

with  $\mathcal{O}(\kappa^{1/4})$  meant in  $L^\infty(0, 1)$ . At the jump points of  $V_B$ , this equation is to be understood in the sense of one-sided limits. We also obtain

$$-2\kappa u_\kappa'' = (h(u_0^2) - h(u_\kappa^2))u_\kappa + \mathcal{O}(\kappa^{1/4}). \quad (30)$$

Now we have on the one hand, in the sense of distributions,

$$\begin{aligned} &((F + \beta_\kappa + V_B + V_\kappa)u_\kappa^2 - K(u_\kappa^2))' \\ &= 2(F + \beta_\kappa + V_B + V_\kappa - k(u_\kappa^2))u_\kappa u_\kappa' + (V_B + V_\kappa)'u_\kappa^2 \\ &= -2\kappa((u_\kappa')^2)' + (V_B + V_\kappa)'u_\kappa^2, \end{aligned}$$

and on the other hand, we have

$$\begin{aligned} &((F + \beta_\kappa + V_B + V_\kappa)u_\kappa^2 - K(u_\kappa^2))' \\ &= \left( (F + \beta_\kappa + V_B + V_\kappa - k(u_\kappa^2))u_\kappa^2 + p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2 \right)' \\ &= \left( -2\kappa u_\kappa'' u_\kappa + p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2 \right)'. \end{aligned}$$

Now let  $[s_0, s_1]$  be a maximal interval of length  $L := s_1 - s_0$  where  $V_B$  is constant, and consider  $x_0, x_1$  with  $s_0 < x_0 < x_1 < s_1$ . Then we have

$$-2\kappa((u_\kappa')^2)' + V_\kappa' u_\kappa^2 = \left( -2\kappa u_\kappa'' u_\kappa + p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2 \right)', \quad \text{on } [x_0, x_1].$$

We integrate over  $[x_0, x_1]$ :

$$\begin{aligned} &-2\kappa(u_\kappa')^2 \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} V_\kappa' u_\kappa^2 dx \\ &= -2\kappa u_\kappa''(x_1)u_\kappa(x_1) + 2\kappa u_\kappa''(x_0)u_\kappa(x_0) + \left( p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2 \right) \Big|_{x_0}^{x_1}. \end{aligned}$$

Now we utilise (30) for  $u_\kappa''(x_0)$  and send  $x_0$  to  $s_0$ :

$$\begin{aligned} &2\kappa(u_\kappa'(s_0))^2 - 2\kappa(u_\kappa'(x_1))^2 + p(u_\kappa^2(s_0)) + \int_{s_0}^{x_1} V_\kappa' u_\kappa^2 dx \\ &= -2\kappa u_\kappa''(x_1)u_\kappa(x_1) - (h(u_0^2(s_0+)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(x_1)) + \mathcal{O}(\kappa^{1/4}), \end{aligned}$$

having recalled that  $u_0^2$  jumps at  $s_0$ . Let us re-arrange this identity into

$$\begin{aligned} &2\kappa(u_\kappa'(s_0))^2 + (h(u_0^2(s_0+)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(s_0)) - p(u_0^2(s_0+)) \\ &= 2\kappa(u_\kappa'(x_1))^2 - 2\kappa u_\kappa''(x_1)u_\kappa(x_1) - \int_{s_0}^{x_1} V_\kappa' u_\kappa^2 dx \\ &\quad + (p(u_\kappa^2(x_1)) - p(u_0^2(x_1))) + (p(u_0^2(x_1)) - p(u_0^2(s_0+))) + \mathcal{O}(\kappa^{1/4}). \end{aligned} \quad (31)$$

We now consider  $x_1$  as a variable in the interval  $J_\kappa := [s_0 + \kappa^{1/4}L, s_0 + 2\kappa^{1/4}L]$ , and we evaluate the  $L^2(J_\kappa)$  norms of both sides of this identity. To this end, we define

$$C_\kappa^+ := -\frac{1}{2} \left( (h(u_0^2(s_0+)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(s_0)) - p(u_0^2(s_0+)) \right). \quad (32)$$

Then we have

$$\begin{aligned} & |\kappa(u'_\kappa(s_0))^2 - C_\kappa^+| \cdot \kappa^{1/8} \\ & \leq \kappa \left\| (u'_\kappa)^2 \right\|_{L^2(J_\kappa)} + \kappa \|u''_\kappa\|_{L^2(J_\kappa)} \cdot \|u_\kappa\|_{L^\infty(J_\kappa)} \\ & \quad + \frac{1}{2} \left\| \int_{s_0}^{(\cdot)} V'_\kappa(s) u_\kappa^2(s) \, ds \right\|_{L^2(J_\kappa)} + \frac{1}{2} \|p(u_\kappa^2(\cdot)) - p(u_0^2(\cdot))\|_{L^2(J_\kappa)} \\ & \quad + \frac{1}{2} \|p(u_0^2(\cdot)) - p(u_0^2(s_0+))\|_{L^2(J_\kappa)} + \mathcal{O}(\kappa^{3/8}). \end{aligned}$$

Now we estimate the terms of the right hand side one after the other. From the inequalities (14) and (16) it follows that

$$\kappa \|(u'_\kappa)^2\|_{L^2(J_\kappa)} \leq \kappa \|u'_\kappa\|_{L^2(J_\kappa)} \|u'_\kappa\|_{L^\infty(J_\kappa)} \leq C\kappa^{3/4}.$$

Next, inequality (15) and the uniform estimates for  $u_\kappa$  show

$$\kappa \|u''_\kappa\|_{L^2(J_\kappa)} \|u_\kappa\|_{L^\infty(J_\kappa)} \leq C\kappa^{1/2}.$$

Since  $\|V'_\kappa\|_{L^\infty(0,1)}$  and  $\|u_\kappa\|_{L^\infty(0,1)}$  are uniformly bounded, it holds

$$\left\| \int_{s_0}^{(\cdot)} V'_\kappa(s) u_\kappa^2(s) \, ds \right\|_{L^2(J_\kappa)} \leq \left( \int_{\kappa^{1/4}L}^{2\kappa^{1/4}L} (Ct)^2 \, dt \right)^{1/2} \leq C\kappa^{3/8}.$$

By Lipschitz continuity of  $p$  on compact subsets of  $(0, \infty)$ , (9) implies

$$\|p(u_\kappa^2(\cdot)) - p(u_0^2(\cdot))\|_{L^2(J_\kappa)} \leq C\kappa^{3/8}.$$

Because  $u_0^2 = h^{-1}(F + V_B(s_0+) + V_0 + \beta_0)$  on  $[s_0, s_0 + L]$ , we also see

$$\|p(u_0^2(\cdot)) - p(u_0^2(s_0+))\|_{L^2(J_\kappa)} \leq C \|V_0(\cdot) - V_0(s_0)\|_{L^2(J_\kappa)} \leq C\kappa^{3/8}.$$

The result then is

$$|\kappa u'_\kappa(s_0)^2 - C_\kappa^+| \leq C\kappa^{1/4}. \quad (33)$$

This does not yet prove the desired lower bound on  $|u'_\kappa(s_0)|$ , because  $C_\kappa^+$  might be very close to zero. However, we can repeat the above reasoning with another interval  $[s_{-1}, s_0]$  on which  $V_B$  is constant, resulting in  $|\kappa u'_\kappa(s_0)^2 - C_\kappa^-| \leq C\kappa^{1/4}$  for a constant

$$C_\kappa^- := -\frac{1}{2} \left( (h(u_0^2(s_0-)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(s_0)) - p(u_0^2(s_0-)) \right). \quad (34)$$



It follows

$$C_\kappa^+ + C_\kappa^- = g_+(u_\kappa^2(s_0)) + g_-(u_\kappa^2(s_0)),$$

where we have introduced functions  $g_\pm: [0, \infty) \rightarrow \mathbb{R}$  defined by

$$g_\pm(z) := \frac{1}{2} \left( (h(z) - h(u_0^2(s_0 \pm)))z + p(u_0^2(s_0 \pm)) - p(z) \right). \quad (35)$$

There is a Taylor expansion hidden in  $g_\pm$ :

$$\begin{aligned} g_\pm(z) &= \frac{1}{2} \left( H(z) - H(u_0^2(s_0 \pm)) - H'(u_0^2(s_0 \pm)) \cdot (z - u_0^2(s_0 \pm)) \right) \\ &= \frac{1}{4} H''(\xi) \cdot (z - u_0^2(s_0 \pm))^2, \end{aligned}$$

with some  $\xi$  between  $z$  and  $u_0^2(s_0 \pm)$ . Then (22) implies

$$\frac{1}{4K_2} (z - u_0^2(s_0 \pm))^2 \leq g_\pm(z) \leq \frac{K_2}{4} (z - u_0^2(s_0 \pm))^2, \quad (36)$$

which brings us to

$$C_\kappa^+ + C_\kappa^- \geq \frac{1}{8K_2} (u_0^2(s_0+) - u_0^2(s_0-))^2.$$

We clearly have

$$|2\kappa u'_\kappa(s_0)^2 - (C_\kappa^+ + C_\kappa^-)| \leq C\kappa^{1/4},$$

which finally yields  $\kappa u'_\kappa(s_0)^2 \geq C_5^2 > 0$  for all  $0 < \kappa < \kappa_0$  and a certain  $C_5$ .

To prove (26), we remark that  $|C_\kappa^+ - C_\kappa^-| \leq C\kappa^{1/4}$ , hence

$$\begin{aligned} \mathcal{O}(\kappa^{1/4}) &\geq \frac{|C_\kappa^+ - C_\kappa^-|}{|h(u_0^2(s_0+)) - h(u_0^2(s_0-))|} \\ &= \frac{1}{2} \left| -u_\kappa^2(s_0) + \frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{h(u_0^2(s_0+)) - h(u_0^2(s_0-))} \right|, \end{aligned}$$

where we have exploited (32) and (34).  $\square$

**Lemma 5.3 (Derivation of the zeroth order asymptotic expansion).** *In the situation of Theorem 1.1, assume that  $V_B$  is piecewise constant and let  $s_0$  be a jump point of  $V_B$ . Assume that  $[s_0, s_0 + L]$  is a maximal interval where  $V_B$  is constant. Then there exist  $C_6 > 0$ ,  $w: \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$\left\| u_\kappa(\cdot) - w\left(\frac{\cdot - s_0}{\kappa^{1/2}}\right) \right\|_{L^\infty([s_0, s_0 + C_6\kappa^{1/2}])} \leq C\kappa^{1/4}. \quad (37)$$

Moreover,  $w$  converges exponentially fast to  $u_0(s_0+)$  for  $y \rightarrow \infty$ ,

$$|w(y) - u_0(s_0+)| + |w'(y)| + |w''(y)| \leq C \exp(-C_7 y), \quad (y \geq 0), \quad (38)$$

so that

$$\left\| w\left(\frac{\cdot}{\kappa^{1/2}}\right) - u_0(s_0+) \right\|_{L^1(0, \infty)} \leq \frac{C}{C_7} \kappa^{1/2}. \quad (39)$$

*Proof.* We rewrite (31): Rename  $x_1$  to  $x \in [s_0, s_0 + 2\kappa^{1/4}L]$ , recall that the left hand side as well as the integral are items of size  $\mathcal{O}(\kappa^{1/4})$ , and apply (30) for  $u''_\kappa(x_1)$ . Then we get

$$2\kappa(u'_\kappa(x))^2 = (h(u_\kappa^2(x)) - h(u_0^2(x)))u_\kappa^2(x) + p(u_0^2(s_0+)) - p(u_\kappa^2(x)) + \mathcal{O}(\kappa^{1/4}).$$

Observe that the Lipschitz continuities of  $h$  and  $u_0$  imply

$$h(u_0^2(s_0+)) - h(u_0^2(x)) = \mathcal{O}(\kappa^{1/4}),$$

hence we find

$$\kappa(u'_\kappa(x))^2 = g_+(u_\kappa^2(x)) + \mathcal{O}(\kappa^{1/4}), \quad s_0 < x \leq s_0 + 2\kappa^{1/4}L. \quad (40)$$

We wish to extract the root here, and therefore we think about the sign of  $u'_\kappa(s_0)$ . From (40) and (36) we learn that  $u'_\kappa(x)$  can change its sign (for the mentioned  $x$ ) only if  $u_\kappa^2(x)$  is near  $u_0^2(s_0+)$ . On the other hand, (9) tells us that  $u_\kappa^2(x) - u_0^2(s_0+) = \mathcal{O}(\kappa^{1/4})$  is a small number for  $x = s_0 + 2\kappa^{1/4}L$ . Hence it is possible to conclude: if  $u_0^2(s_0-) < u_0^2(s_0+)$ , then  $u'_\kappa(s_0) > 0$ , and *vice versa*. Without loss of generality, we assume this case. From Remark 5.2, (25) and (36) we then also find that  $u_\kappa^2(s_0) \leq u_0^2(s_0+) - c$  holds for some positive  $c$ . Next we get

$$\kappa^{1/2}u'_\kappa(x) = \sqrt{g_+(u_\kappa^2(x))} + \mathcal{O}(\kappa^{1/4}), \quad s_0 < x \leq s_0 + C_6\kappa^{1/2},$$

and  $C_6$  will be chosen later in such a way that  $g_+(u_\kappa^2(x)) \geq c > 0$  is ensured for the mentioned  $x$  and some small constant  $c$ , making the manipulation on the right hand side valid.

Introducing the variable transformation  $y = \frac{1}{\kappa^{1/2}}(x - s_0)$ , we consider the initial value problem for a function  $w$ ,

$$\begin{cases} w'(y) = +\sqrt{g_+(w^2(y))}, & 0 < y < \infty, \\ w(0) = \frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{V_B(s_0+) - V_B(s_0-)}, \end{cases} \quad (41)$$

compare (26). The life span of  $w$  is *a priori* not known, but the constant function  $\hat{w}(y) := u_0(s_0+)$  solves the same differential equation and has an initial value  $\hat{w}(0) > w(0)$ , hence the uniqueness principle gives  $w(y) < \hat{w}(y)$ , making a blowup of  $w$  impossible. The classical theory of upper and lower solutions (c.f. [21, II §9–IV]) can be applied: Let  $\underline{w}$  and  $\bar{w}$  be functions solving

$$\begin{aligned} \underline{w}'(y) &= \frac{1}{2\sqrt{K_2}} |\underline{w}^2(y) - u_0^2(s_0+)|, & \underline{w}(0) &= w(0), \\ \bar{w}'(y) &= \frac{\sqrt{K_2}}{2} |\bar{w}^2(y) - u_0^2(s_0+)|, & \bar{w}(0) &= w(0), \end{aligned}$$

compare (36). Then  $\underline{w}(y) \leq w(y) \leq \bar{w}(y) < \hat{w}(y)$ , for  $0 \leq y < \infty$ , and in particular, (38) follows (using (41) for estimating  $w'$  and  $w''$ ).

Now we pick a small number  $c > 0$ , determine  $C_6$  by  $g_+(\bar{w}^2(y)) > 2c$  on  $[0, C_6]$ , and classical perturbation arguments then show (37).  $\square$

**Remark 5.4.** *Let us summarize what we have obtained so far: (9) provides us the starting term of an asymptotic expansion of  $u_\kappa(x)$ , valid for points  $x$  whose distance to the nearest jump point of  $V_B$  is at least  $\mathcal{O}(\kappa^{1/4})$ . On the other hand, (37) takes care of those  $x$  whose distance to the nearest jump point is at most  $\mathcal{O}(\kappa^{1/2})$ . Hence a gap remains between both regions, and this gap is handled in the next lemma. The bound  $\kappa^{1/8}$  in (42) will be improved in Section 6 to  $\kappa^{1/4}$ .*

**Lemma 5.5 (Preliminary estimates for the zeroth order asymptotic expansion).** *Let  $w$  be the function constructed in Lemma 5.3 for a jump point  $s_0$  of  $V_B$ . Let  $[s_0, s_0 + L]$  be a maximal interval where  $V_B$  is constant. Then*

$$\left\| u_\kappa(\cdot) - \frac{u_0(\cdot)}{u_0(s_0+)} w\left(\frac{\cdot - s_0}{\kappa^{1/2}}\right) \right\|_{L^\infty(s_0, s_0+L/2)} \leq C\kappa^{1/8}, \quad 0 < \kappa < \kappa_0. \quad (42)$$

*A similar estimate is valid in a left neighbourhood of the jump point  $s_0$ .*

*Then there are open disjoint neighbourhoods  $\Omega_i$  of the jump points  $s_i$  of  $V_B$ , there is a function  $c_0: [0, 1] \rightarrow (0, \infty)$  and a family of functions  $W_\kappa$  such that:*

$$\begin{aligned} W_\kappa &\in C^1([0, 1]; \mathbb{R}) \quad \text{and} \quad W_\kappa \text{ has piecewise } C^2 \text{ regularity,} \\ c_0 &\text{ has piecewise } C^2 \text{ regularity,} \\ c_0(x) &= u_0(s_i \pm) \text{ in } \Omega_i, \\ W_\kappa(x) &= w\left(\frac{x - s_i}{\kappa^{1/2}}\right) \text{ in } \Omega_i, \\ 2\kappa W_\kappa''(x) &= W_\kappa(x) (h(W_\kappa^2(x)) - h(u_0^2(s_i \pm))) \text{ in } \Omega_i, \\ |W_\kappa''(x)| &\leq C \text{ outside } \cup_i \Omega_i, \end{aligned} \quad (43)$$

$$\begin{aligned} \left\| \frac{W_\kappa}{c_0} - 1 \right\|_{L^1(0,1)} &\leq C\kappa^{1/2}, \\ \left\| u_\kappa - u_0 \frac{W_\kappa}{c_0} \right\|_{L^\infty(0,1)} &\leq C\kappa^{1/8}. \end{aligned} \quad (44)$$

*Proof.* Without loss of generality, we assume  $u_0^2(s_0-) < u_0^2(s_0+)$ , which corresponds to  $u'_\kappa(s_0) > 0$ ,  $u_\kappa^2(s_0) < u_0^2(s_0+)$  and  $w'(y) > 0$  everywhere. By construction of  $w$ , it is easily seen that estimate (42) even holds with the better rate  $\kappa^{1/4}$  in the regimes  $[s_0, s_0 + C_6\kappa^{1/2}]$  and  $[s_0 + \kappa^{1/4}L, s_0 + L/2]$ . Now we treat the remaining part  $(s_0 + C_6\kappa^{1/2}, s_0 + \kappa^{1/4}L)$ , and our first step is to think about how large can  $u_\kappa^2(x)$  be on the interval  $J_\kappa := [s_0, s_0 + \kappa^{1/4}L]$ . The maximum can not be attained on the left endpoint  $s_0$ , by assumption. If the maximum is attained at the right endpoint, then (9) and the Lipschitz continuity of  $u_0$  imply  $\max_{J_\kappa} u_\kappa^2(x) \leq u_0^2(s_0+) + C\kappa^{1/4}$ . And if the maximum is attained inside of  $J_\kappa$ , then (40) and (36) yield

$$\max_{J_\kappa} u_\kappa^2(x) \leq u_0^2(s_0+) + C\kappa^{1/8}. \quad (45)$$

After this preparation, we now consider the local extrema of the function

$$d(x) := u_\kappa(x) - w\left(\frac{x - s_0}{\kappa^{1/2}}\right), \quad x \in [s_0 + C_6\kappa^{1/2}, s_0 + \kappa^{1/4}L].$$

We know that  $|d| = \mathcal{O}(\kappa^{1/4})$  at the endpoints of this interval, and if  $x_*$  is an interior local extremum of  $d$ , then  $u_* := u_\kappa(x_*)$  and  $w_* := w((x_* - s_0)/\kappa^{1/2})$  satisfy

$$g_+(u_*^2) + \mathcal{O}(\kappa^{1/4}) = g_+(w_*^2), \quad (46)$$

by (40) and (41). Put  $c_0 := u_0(s_0+)$ . Now we distinguish the cases  $u_* \leq c_0$  and  $u_* > c_0$ , and we remark that the inequality  $w_* < c_0$  holds in both of them.

Assume  $u_* \leq c_0$ . Without loss of generality, we also suppose  $w_* \leq u_*$ . Noticing that  $g_+$  is monotonically decreasing on  $[w_*^2, c_0^2]$  and convex there, a Taylor expansion,  $g_+''(z) = h'(z)/2$ , and (22) then tell us

$$\begin{aligned} C\kappa^{1/4} &\geq g_+(w_*^2) - g_+(u_*^2) = g_+'(u_*^2) \cdot (w_*^2 - u_*^2) + \frac{1}{2}g_+''(\xi) \cdot (w_*^2 - u_*^2)^2 \\ &\geq 0 + \frac{1}{4K_2}(w_*^2 - u_*^2)^2, \end{aligned}$$

which settles the first case  $u_* \leq c_0$ .

We come to the slightly harder case  $u_* > c_0$ , in which we clearly have

$$\|d\|_{L^\infty(s_0 + C_6\kappa^{1/2}, s_0 + \kappa^{1/4}L)} = |u_* - c_0| + |c_0 - w_*|,$$

and the first item on the right is bounded by  $C\kappa^{1/8}$ , from (45). From this first term estimate, (46), and twice (36) we then deduce that

$$\begin{aligned} C\kappa^{1/4} &\geq \frac{K_2}{4}(u_*^2 - c_0^2)^2 \geq g_+(u_*^2) \geq g_+(w_*^2) - \mathcal{O}(\kappa^{1/4}) \\ &\geq \frac{1}{4K_2}(w_*^2 - c_0^2)^2 - \mathcal{O}(\kappa^{1/4}), \end{aligned}$$

which brings us to  $|c_0 - w_*| \leq C\kappa^{1/8}$ . This concludes the second case, and (42) is readily seen.

Now we match the various asymptotic expansions of  $u_\kappa$ . Let  $\{s_2, s_3, \dots, s_N\} \subset (0, 1)$  be the jump points of  $V_B$ , increasingly ordered, and define  $s_1 := 0$ ,  $s_{N+1} := 1$ . Let  $\Omega_1, \dots, \Omega_{N+1}$  be disjoint open neighbourhoods of  $s_1, \dots, s_{N+1}$ . For  $i = 2, \dots, N$ , let  $w_{+,i}$  and  $w_{-,i}$  be the layer profiles near  $s_i$ , defined similarly to (41). Observe that their derivatives match,  $w'_{-,i}(0) = w'_{+,i}(0)$ , by the very choice of  $w_{\pm,i}(0)$ . Then we construct a function  $\tilde{w}_\kappa^i \in C^1(\mathbb{R})$  that makes the transition between  $u_0(s_i-)$  and  $u_0(s_i+)$  in  $\Omega_i$  (having only an exponentially small error at the endpoints of  $\Omega_i$ ):

$$\tilde{w}_\kappa^i(x) := \begin{cases} w_{-,i}\left(\frac{x - s_i}{\kappa^{1/2}}\right) & : x \leq s_i, \\ w_{+,i}\left(\frac{x - s_i}{\kappa^{1/2}}\right) & : x \geq s_i. \end{cases}$$

Outside the jump point set,  $\tilde{w}_\kappa^i$  has  $C^2$  regularity. We also define  $\tilde{w}_\kappa^1 \equiv u_0(0)$  and  $\tilde{w}_\kappa^{N+1} \equiv u_0(1)$  as constant functions. Next we choose a partition of unity: Define  $s_0 := -1$ ,  $s_{N+2} := 2$ , and select functions  $(\varphi_i)_{i=1,\dots,N+1} \subset C^\infty(\mathbb{R}, [0, 1])$  such that  $\text{supp } \varphi_i \subseteq (s_{i-1}, s_{i+1})$ ,  $\varphi_i \equiv 1$  in a  $\Omega_i$ , where  $i = 1, \dots, N+1$ ; and  $\sum_{j=1}^{N+1} \varphi_j(x) = 1$  on  $[0, 1]$ . Then the function  $W_\kappa$  is given as

$$W_\kappa(x) := \sum_{i=1}^{N+1} \tilde{w}_\kappa^i(x) \varphi_i(x).$$

Moreover, define the piecewise constant functions  $\tilde{c}_0^1 \equiv u_0(0)$ ,  $\tilde{c}_0^{N+1} \equiv u_0(1)$  and  $\tilde{c}_0^i = \chi_{(-\infty, s_i]} u_0(s_i-) + \chi_{(s_i, \infty)} u_0(s_i+)$  for  $i = 2, \dots, N$ . Finally, we put

$$\widetilde{W}_\kappa(x) := \sum_{i=1}^{N+1} \frac{\tilde{w}_\kappa^i(x)}{\tilde{c}_0^i(x)} \varphi_i(x), \quad c_0(x) := \frac{W_\kappa(x)}{\widetilde{W}_\kappa(x)}.$$

The inequality (44) follows from (39), and (43) is easily checked. We remark

$$\left| \widetilde{W}_\kappa(x) - 1 \right| \leq \sum_{i=1}^{N+1} \frac{|\tilde{w}_\kappa^i(x) - \tilde{c}_0^i(x)|}{\tilde{c}_0^i(x)} \varphi_i(x) \leq C \exp\left(-C\kappa^{-1/2}\right), \quad (47)$$

valid for  $x \notin \cup_i \Omega_i$ .  $\square$

## 6. Refined remainder estimates

We complete the proof of Theorem 2.2 by considering differential equations to the remainder terms

$$R_{u_\kappa} := u_\kappa - \frac{u_0}{c_0} W_\kappa, \quad R_{V_\kappa} := V_\kappa - V_0, \quad R_{\beta_\kappa} := \beta_\kappa - \beta_0, \quad R_{V_\kappa, \beta_\kappa} := R_{V_\kappa} + R_{\beta_\kappa}.$$

Then  $R_{V_\kappa}$  and  $R_{u_\kappa}$  solve the differential equations

$$\lambda^2 R_{V_\kappa}'' = 2u_\kappa R_{u_\kappa}^2 - R_{u_\kappa}^2 + u_0^2 \left( \frac{W_\kappa^2}{c_0^2} - 1 \right), \quad (48)$$

$$\begin{aligned} & \frac{2\kappa}{W_\kappa} \left( W_\kappa^2 \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \right)' \\ &= -R_{V_\kappa, \beta_\kappa} u_\kappa - (h(u_0^2) - k(u_\kappa^2)) u_\kappa - 2\kappa W_\kappa'' \frac{u_\kappa}{W_\kappa} - \frac{2\kappa}{W_\kappa} \left( W_\kappa^2 \left( \frac{u_0}{c_0} \right)' \right)', \end{aligned} \quad (49)$$

where we have recalled (5) and (29).

*Proof of Theorem 2.2.* We will show

$$\kappa \left\| W_\kappa \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \right\|_{L^2(0,1)}^2 + \|R_{V_\kappa}'\|_{L^2(0,1)}^2 + \|R_{u_\kappa}\|_{L^2(0,1)}^2 \leq C\kappa, \quad (50)$$

from which we obtain estimate (10) by Poincaré's inequality. Estimate (11) follows by interpolation of the inequalities  $\left\| \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \right\|_{L^2(0,1)} \leq C$  and  $\left\| \frac{R_{u_\kappa}}{W_\kappa} \right\|_{L^2(0,1)} \leq C\kappa^{1/2}$ , as  $W_\kappa$  is uniformly bounded from above and away from zero.

Multiplying equation (49) by  $R_{u_\kappa}$  and integrating by parts we obtain

$$\begin{aligned} 2\kappa \int_0^1 W_\kappa^2 \left( \frac{R_{u_\kappa}}{W_\kappa} \right)'^2 dx &= \int_0^1 R_{V_\kappa, \beta_\kappa} R_{u_\kappa} u_\kappa dx + \int_0^1 (h(u_0^2) - k(u_\kappa^2)) u_\kappa R_{u_\kappa} dx \\ &\quad + 2\kappa \int_0^1 W_\kappa'' \frac{u_\kappa}{W_\kappa} R_{u_\kappa} dx + 2\kappa \int_0^1 W_\kappa \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \cdot W_\kappa \left( \frac{u_0}{c_0} \right)' dx. \end{aligned} \quad (51)$$

By Young's inequality it follows that

$$2\kappa \int_0^1 W_\kappa \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \cdot W_\kappa \left( \frac{u_0}{c_0} \right)' dx \leq \kappa \left\| W_\kappa \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \right\|_{L^2(0,1)}^2 + C\kappa. \quad (52)$$

Concerning  $R_{\beta_\kappa}$ , observe that (5) and (29) give us the representation

$$R_{\beta_\kappa} = -R_{V_\kappa} - (h(u_0^2) - k(u_\kappa^2)) - 2\kappa \frac{u_\kappa''}{u_\kappa}.$$

Let  $I = [s_1, s_2]$  be an interval of length  $L$ , on which  $V_B$  is constant, and  $I_\kappa := [s_1 + \kappa^{1/4}L, s_2 - \kappa^{1/4}L]$ . Then, due to (15),

$$\begin{aligned} |R_{\beta_\kappa}| &\leq \frac{2}{L} \|R_{\beta_\kappa}\|_{L^1(I_\kappa)} \\ &\leq C \left( \|R_{V_\kappa}\|_{L^1(I_\kappa)} + \|h(u_\kappa^2) - h(u_0^2)\|_{L^1(I_\kappa)} + \kappa \|\ln(u_\kappa^2)\|_{L^1(I_\kappa)} + \kappa \left\| \frac{u_\kappa''}{u_\kappa} \right\|_{L^1(I_\kappa)} \right) \\ &\leq C \|R_{V_\kappa}\|_{L^2(0,1)} + C \|R_{u_\kappa}\|_{L^2(0,1)} + C\kappa^{1/2}. \end{aligned}$$

Employing (48) one finds

$$\begin{aligned} \frac{\lambda^2}{2} \int_0^1 R_{V_\kappa, \beta_\kappa}^2 dx &= - \int_0^1 R_{V_\kappa, \beta_\kappa} R_{u_\kappa} u_\kappa dx + \frac{1}{2} \int_0^1 R_{u_\kappa}^2 R_{V_\kappa, \beta_\kappa} dx \\ &\quad - \frac{1}{2} \int_0^1 u_0^2 \left( \frac{W_\kappa^2}{c_0^2} - 1 \right) R_{V_\kappa, \beta_\kappa} dx \\ &\leq - \int_0^1 R_{V_\kappa, \beta_\kappa} R_{u_\kappa} u_\kappa dx + \frac{1}{2} \|R_{V_\kappa, \beta_\kappa}\|_{L^\infty(0,1)} \cdot \|R_{u_\kappa}\|_{L^2(0,1)}^2 \\ &\quad + C \|R_{V_\kappa, \beta_\kappa}\|_{L^\infty(0,1)} \left\| \frac{W_\kappa}{c_0} - 1 \right\|_{L^1(0,1)} \end{aligned} \quad (53)$$

$$\leq - \int_0^1 R_{V_\kappa, \beta_\kappa} R_{u_\kappa} u_\kappa \, dx + C\kappa^{1/4} \|R_{u_\kappa}\|_{L^2(0,1)}^2 + C\kappa^{1/2} \|R_{V_\kappa, \beta_\kappa}\|_{L^\infty(0,1)},$$

using (6), (7), (44). The Sobolev embedding  $H^1(0,1) \hookrightarrow L^\infty(0,1)$  and Poincaré's inequality give us

$$\begin{aligned} C\kappa^{1/2} \|R_{V_\kappa, \beta_\kappa}\|_{L^\infty(0,1)} &\leq C\kappa^{1/2} \|R'_{V_\kappa}\|_{L^2(0,1)} + C\kappa^{1/2} \|R_{u_\kappa}\|_{L^2(0,1)} + C\kappa \quad (54) \\ &\leq \frac{\lambda^2}{4} \|R'_{V_\kappa}\|_{L^2(0,1)}^2 + C\kappa^{1/2} \|R_{u_\kappa}\|_{L^2(0,1)} + C\kappa \end{aligned}$$

Now we add the inequalities (51), (52), and (53), and we bring (54) into play:

$$\begin{aligned} &\kappa \left\| W_\kappa \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \right\|_{L^2(0,1)}^2 + \frac{\lambda^2}{4} \|R'_{V_\kappa}\|_{L^2(0,1)}^2 \\ &\leq \int_0^1 \left( (h(u_0^2) - h(u_\kappa^2)) + \frac{2\kappa W''_\kappa}{W_\kappa} \right) u_\kappa R_{u_\kappa} \, dx \\ &\quad + C\kappa + C\kappa^{1/4} \|R_{u_\kappa}\|_{L^2(0,1)}^2 + C\kappa^{1/2} \|R_{u_\kappa}\|_{L^2(0,1)}. \end{aligned}$$

To discuss the integral on the right hand side, we distinguish the cases  $x \in \Omega_i$  and  $x \notin \cup_i \Omega_i$ .

Suppose  $x$  to be in the right part of  $\Omega_i$ . Then we conclude from (43) that

$$\begin{aligned} &(h(u_0^2) - h(u_\kappa^2))(x) + \frac{2\kappa W''_\kappa(x)}{W_\kappa(x)} \\ &= h(u_0^2(x)) - h(u_\kappa^2(x)) + h(W_\kappa^2(x)) - h(u_0^2(s_i+)) \\ &= \left( h \left( W_\kappa^2(x) \frac{u_0^2(x)}{c_0^2(x)} \right) - h(u_\kappa^2(x)) \right) \\ &\quad + \left( h(W_\kappa^2(x)) - h(u_0^2(s_i+)) - h \left( \frac{W_\kappa^2(x)}{u_0^2(s_i+)} u_0^2(x) \right) + h(u_0^2(x)) \right) \\ &=: T_1(x) + T_2(x). \end{aligned}$$

Here we have used  $c_0(x) = u_0(s_i+)$  for these  $x$ . By monotonicity of  $h$ , we have

$$\begin{aligned} T_1(x) u_\kappa(x) R_{u_\kappa}(x) &= \frac{u_\kappa(x)}{u_\kappa(x) + \frac{u_0(x)}{c_0(x)} W_\kappa(x)} T_1(x) \left( u_\kappa^2(x) - \frac{u_0^2(x)}{c_0^2(x)} W_\kappa^2(x) \right) \\ &\leq -\frac{1}{K_2} \frac{u_\kappa(x)}{u_\kappa(x) + \frac{u_0(x)}{c_0(x)} W_\kappa(x)} \left( u_\kappa^2(x) - \frac{u_0^2(x)}{c_0^2(x)} W_\kappa^2(x) \right)^2 \\ &\leq -C_8 R_{u_\kappa}^2(x), \end{aligned}$$

for a certain positive  $C_8$ , and this estimate is even valid for all  $x \in [0, 1]$ .

The term  $T_2(x)$  can be handled using Lemma Appendix A.1:

$$\begin{aligned}
|T_2(x)| &\leq C \left| \ln \frac{u_0^2(x)}{u_0^2(s_i+)} \right| \cdot \left| \ln \frac{W_\kappa^2(x)}{u_0^2(s_i+)} \right| \\
&\leq C |u_0(x) - u_0(s_i+)| \cdot |W_\kappa(x) - u_0(s_i+)| \\
&\leq C \kappa^{1/2} \frac{x - s_i}{\kappa^{1/2}} \cdot \exp \left( -C_7 \frac{x - s_i}{\kappa^{1/2}} \right),
\end{aligned}$$

which implies  $\sup_{\Omega_i} |T_2(x)| \leq C \kappa^{1/2}$ .

Next we handle the case  $x \notin \cup_i \Omega_i$ . Then we calculate as follows:

$$\begin{aligned}
\left| \frac{2\kappa W_\kappa''(x)}{W_\kappa(x)} \right| &\leq C \kappa, \\
h(u_0^2(x)) - h(u_\kappa^2(x)) &= T_1(x) + h(u_0^2(x)) - h \left( W_\kappa^2(x) \frac{u_0^2(x)}{c_0^2(x)} \right) \\
&= T_1(x) + h(u_0^2(x)) - h \left( \widetilde{W}_\kappa^2(x) u_0^2(x) \right) \\
&= T_1(x) + \mathcal{O}(\kappa),
\end{aligned}$$

by (47). Summing up what we have obtained so far, we get

$$\begin{aligned}
&\kappa \left\| W_\kappa \left( \frac{R_{u_\kappa}}{W_\kappa} \right)' \right\|_{L^2(0,1)}^2 + \frac{\lambda^2}{4} \|R'_{V_\kappa}\|_{L^2(0,1)}^2 \\
&\leq -C_8 \|R_{u_\kappa}\|_{L^2(0,1)}^2 + C \kappa + C \kappa^{1/4} \|R_{u_\kappa}\|_{L^2(0,1)}^2 + C \kappa^{1/2} \|R_{u_\kappa}\|_{L^2(0,1)},
\end{aligned}$$

and now (50) is deduced by Young's inequality.  $\square$

## Appendix A. Appendix

We conclude this paper with a useful tiny technical lemma.

**Lemma Appendix A.1.** *Let  $I \subset (0, \infty)$  be a compact interval, and  $h: I \rightarrow \mathbb{R}$  be twice continuously differentiable, and define  $C_h := \max_I |h''(t)t^2 + h'(t)t|$ . If  $p, q, \lambda$  are positive numbers with  $p, q, \lambda p, \lambda q \in I$ , then*

$$|h(\lambda p) - h(p) - h(\lambda q) + h(q)| \leq C_h \left| \ln \frac{q}{p} \right| \cdot |\ln \lambda|.$$

*Proof.* We define a function  $g$  by

$$h(t) =: g(\ln t), \quad t \in I,$$

and we observe that  $g''(\tau) = h''(t)t^2 + h'(t)t$  for  $t = e^\tau \in I$ . Now it suffices to



calculate

$$\begin{aligned}
h(\lambda p) - h(p) - h(\lambda q) + h(q) &= g(\ln \lambda + \ln p) - g(\ln p) - g(\ln \lambda + \ln q) + g(\ln q) \\
&= \int_{\tau=0}^{\ln \lambda} g'(\tau + \ln p) - g'(\tau + \ln q) \, d\tau \\
&= \int_{\tau=0}^{\ln \lambda} \int_{\sigma=\ln q}^{\ln p} g''(\tau + \sigma) \, d\sigma \, d\tau.
\end{aligned}$$

□

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